Universität Tübingen
Mathematisches Institut
Prof. Dr. Christian Lubich

Tübingen, December 2, 2025

7. Sheet for Numerics of Stationary Differential Equations (with Solutions)

Exercise 19

We define: $u \in L^2(\Omega)$ admits the *weak derivative* $\partial_i u$ (for $i = 1, \dots, n$), if $\partial_i u \in L^2(\Omega)$ and

$$(\phi, \partial_i u)_0 = -\left(\frac{\partial \phi}{\partial x_i}, u\right)_0 \qquad \text{for all } \phi \in C_0^\infty(\overline{\Omega}).$$

Show for bounded piecewise C^1 domains Ω:

(a) For $u \in C^1(\overline{\Omega})$ the classical derivative $\partial u / \partial x_i$ is also a weak derivative.

(b) For $u \in H^1(\Omega)$ the generalized derivatives (from the lecture) are also weak derivatives.

It holds (you do not need to prove this): If the weak derivatives of $u \in L^2(\Omega)$ exist, they are also generalized derivatives and therefore $u \in H^1(\Omega)$.

Exercise 20

Assume a triangulation of a bounded domain $\Omega \subset \mathbb{R}^2$. Let u be a function which is C^1 on each triangle.

Show:

$$u \in H^1(\Omega) \iff u \in C(\overline{\Omega})$$

Hint: $u \in H^1(\Omega) \iff u \in L^2(\Omega)$ and u admits weak derivatives (see previous exercise).

Exercise 21

(a) Give a continuous function on $[0, 1]$ which is not an element of $H^1(0, 1)$.

(b) Let Ω be a ball in $\mathbb{R}^3$ with center at the origin. Show: For $\alpha < 1/2$, the function $u(x) = \|x\|^{-\alpha}$ is in $H^1(\Omega)$.

Exercise 22

Let $\Omega = [a, b]$ be a real interval. Show: $H^1(a, b) \subset C[a, b]$.

Hint:

(a) Show: $|v(x)| \leq C\|v\|_1$ for $v \in C^\infty[a, b]$.

(b) Use the density of C^∞ in H^1 with respect to the $\| \cdot \|_1$-norm.

Solutions are discussed on Tuesday December 9, 2025
Tutor: Georgios Vretinaris - if you have question just come to my office (C3P16) or write me an email.