Universität Tübingen
Mathematisches Institut
Prof. Dr. Christian Lubich

Tübingen, November 25, 2025

6. Sheet for Numerics of Stationary Differential Equations (with Solutions)

Exercise 16

Consider the Helmholtz equation with Neumann boundary conditions:

$$-\Delta u + u = f \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = g \quad \text{on } \Gamma. \tag{$**$}$$

Show for $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ that the following statements are equivalent:

(a) u is a solution to $(**)$.

(b) It holds that

$$\int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + uv \right) d(x,y) = \int_\Omega fv\, d(x,y) + \int_\Gamma gv\, d\sigma$$

for all $v \in C^1(\Omega) \cap C(\overline{\Omega})$.

(c) u is a solution to the variational problem

$$\frac{1}{2}\int_\Omega \left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + v^2 \right] d(x,y) - \int_\Omega fv\, d(x,y) - \int_\Gamma gv\, d\sigma = \text{min!}$$

under all $v \in C^1(\Omega) \cap C(\overline{\Omega})$.

Exercise 17

Consider the Poisson equation with mixed boundary conditions:

$$-\Delta u = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma_0, \qquad \frac{\partial u}{\partial n} = g \quad \text{on } \Gamma \setminus \Gamma_0.$$

Analogously to the previous exercise, state the corresponding variational problem and show the equivalence of the statements.

Exercise 18

Let V, W be normed vector spaces and let $L : V \to W$ be a linear map. Show:

$$L \text{ continuous} \iff L \text{ continuous at } 0 \iff L \text{ bounded.}$$

Solutions are discussed on Tuesday December 2, 2025
Tutor: Georgios Vretinaris - if you have question just come to my office (C3P16) or write me an email.