

3. Sheet for Numerics of Stationary Differential Equations (with Solutions)

Exercise 9

Given is the parameter-dependent differential equation

$$y'(t) = f(t, y(t), p).$$

Show that

$$P(t) = \frac{\partial}{\partial p} y(t|t_0, x_0, p)$$

is the solution of the inhomogeneous, linear matrix differential equation

$$P'(t) = C(t)P(t) + B(t),$$

where

$$C(t) = \frac{\partial f}{\partial y}(t, y(t|t_0, x_0, p), p) \quad \text{and} \quad B(t) = \frac{\partial f}{\partial p}(t, y(t|t_0, x_0, p), p)$$

with $P(t_0) = 0$.

Exercise 10

If one approximates in the variational problem

$$\int_a^b f(t, y, y') dt = \min!$$

the integral through the trapezoidal rule and the derivatives through a difference quotient, one gets the following minimization problem

$$\frac{h}{2} f\left(a, y(a), \frac{y(a+h) - y(a)}{h}\right) + h \sum_{i=1}^{n-1} f\left(t_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right) + \frac{h}{2} f\left(b, y(b), \frac{y(b) - y(b-h)}{h}\right) = \min!$$

Show for given boundary values that the solution of this problem is the same as if one applies the midpoint rule to the Euler–Lagrange equations

$$y' = v, \quad p' = \frac{\partial f}{\partial y}(t, y, v), \quad 0 = p - \frac{\partial f}{\partial y'}(t, y, v).$$

Hint: The derivative in the midpoint rule is replaced by a difference quotient of the form

$$y'(t_i) \rightarrow \frac{y_{i+1} - y_{i-1}}{2h}$$

(analogous for p').

Exercise 11

Formulate and prove a 2-dimensional version of the fundamental lemma of calculus of variations.

Exercise 12

Derive the Euler–Lagrange equations for the minimal surface problem

$$\int_{\Omega} \sqrt{1 + u_x^2(x, y) + u_y^2(x, y)} \, d(x, y) = \min!$$

with given values on the boundary $u = g$ on $\partial\Omega$. Here Ω is a bounded domain with smooth boundary in \mathbb{R}^2 .

Solutions are discussed on Tuesday November 19, 2025

Tutor: Georgios Vretinaris - if you have question just come to my office (C3P16) or write me an email.