Universität Tübingen
Mathematisches Institut
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Tübingen, October 21, 2025

1. Exercise Sheet for Numerics of Stationary Differential Equations

Exercise 1:

Let $R(\cdot,\cdot)$ be the resolvent of the linear differential equation $\dot{y}(t) = C(t)\tilde{y}(t)$, with $C(t) \in \mathbb{R}^{d\times d}$. Show:

(a) For fixed t_0, $R(\cdot, t_0)$ is the solution of the problem

$$\frac{d}{dt} R(t,t_0) = C(t)R(t,t_0), \quad R(t_0,t_0) = I.$$

(b) The solution of the inhomogeneous initial value problem

$$y'(t) = C(t)y(t) + q(t), \quad y(t_0) = y_0 \in \mathbb{R}^d$$

is given by

$$y(t) = R(t,t_0)y_0 + \int_{t_0}^{t} R(t,s)q(s)\,ds.$$

Exercise 2:

For the boundary value problem

$$y' = C(t)y + q(t), \qquad Ay(a) + By(b) = 0$$

consider the *sensitivity matrix*

$$E(t) = AR(a,t) + BR(b,t), \qquad (a \le t \le b).$$

(a) Show: $E(t)$ is invertible for all $t \in [a,b] \iff E(t)$ is invertible for one $t \in [a,b]$.
This is fulfilled in the following.

(b) Show: The unique solution of the above boundary value problem is given by

$$y(t) = \int_a^b G(t,s)q(s)\,ds$$

with the *Green's function*

$$G(t,s) = \begin{cases} E(t)^{-1}AR(a,s), & \text{for } a \le s \le t \le b, \\ -E(t)^{-1}BR(b,s), & \text{for } a \le t \le s \le b. \end{cases}$$

Hint: Represent $y(t)$ as the sum of the solution $v(t)$ of the associated initial value problem with initial value $v(a) = 0$ and the solution $w(t)$ of the associated homogeneous initial value problem with suitable initial value $w(0) = w_0$.

(c) (Sensitivity against perturbations of the inhomogeneity)
Let $y, \tilde{y}$ be the solutions of the boundary value problem

$$y' = C(t)y + q(t), \quad Ay(a) + By(b) = r,$$
$$\tilde{y}' = C(t)\tilde{y} + \tilde{q}(t), \quad A\tilde{y}(a) + B\tilde{y}(b) = r.$$

Show:

$$\max_{a\le t\le b} \|y(t) - \tilde{y}(t)\| \le \gamma \max_{a\le t\le b} \|q(t) - \tilde{q}(t)\|$$

with

$$\gamma = \max_{a\le s\le b} \int_a^b \|G(t,s)\|\,ds \le (b-a) \max_{a\le s,t\le b} \|G(t,s)\|.$$

Exercise 3:

(a) Reformulate the initial value problem (with real parameter $\lambda \neq 0$)

$$u'' = \lambda^2 u, \qquad u(0) = 0, \quad u(1) = 1$$

into a 1st order system by introducing $v = u'/\lambda$. Calculate its resolvent and the Green's function of the boundary value problem. Prove that for $\lambda \to +\infty$ the resolvent grows like e^{λ}, whereas the Green's function remains limited independently of λ.

(I.e. the initial value problem is ill-conditioned, the boundary value problem is well conditioned.)

(b) For which values of $\omega \in \mathbb{R}$ is the boundary value problem

$$u'' = -\omega^2 u, \qquad u(0) = 0, \quad u(1) = 1$$

uniquely solvable? How do the resolvent of the initial value problem and the Green's function of the boundary value problem behave for $\omega \to \pi$?

(Initial value problem well-conditioned, boundary value problem ill-conditioned.)

Hints: $R(t,s) = e^{C(t-s)}$, diagonalize C. $\lambda = i\omega$ in (b) saves computational effort.

Solutions are discussed on Tuesday October 28, 2025
Tutor: Georgios Vretinaris - if you have question just come to my office (C3P16) or write me an email.