Lecture 4: The Black–Scholes Equation and Option Pricing

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Today's Agenda

1. The Black–Scholes equation: motivation and setup.
2. Geometric Brownian motion: motivation and properties.
3. Moments, variance, and log-normality of asset prices.
4. Rigorous derivation of the Black–Scholes PDE.
5. Terminal and boundary conditions; Black–Scholes formulas.

1. The Black–Scholes Equation: Introduction and Goal

References: [BK04, GJ10, Sey09]
Goal: Find equations to determine the value of an option on a single underlying asset. Throughout this chapter, we make the assumptions (A1)–(A5) from Section 1.3 unless otherwise stated.

2. Geometric Brownian Motion

2.1 Motivating the Model of the Underlying Asset

First step: Model the price of the underlying by a suitable process S_t.
For the value of a bond with interest rate $r > 0$, we have

$$B_t = B_0 e^{rt}$$

We want to "stochastify" this equation with a Wiener process in order to model the underlying (risky) asset.

First attempt:

$$S_t = S_0 e^{at} + \sigma W_t, \qquad a, \sigma \in \mathbb{R}$$

Problem: S_t can take negative values, which is not realistic for asset prices.

Second attempt: Notice for the bond:

$$\ln B_t = \ln B_0 + rt$$

This motivates the ansatz:

$$\ln S_t = \ln S_0 + at + \sigma W_t$$

for some $a, \sigma \in \mathbb{R}$ to model the underlying. The parameter σ is called the **volatility**. Applying the exponential function gives:

$$S_t = S_0 \exp(at + \sigma W_t)$$

so $S_t \geq 0$ if $S_0 \geq 0$. In fact, S_t is the *geometric Brownian motion* from Section 2.4, and solves the SDE:

$$dS_t = aS_t dt + \sigma S_t dW_t$$

with S_0 as initial value.
Let us relate this to the more familiar notation:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

with $\mu = a + \sigma^2/2$. The parameter μ is the **expected rate of return** including drift and the Itô correction.

Interpretation:

$$\frac{dS_t}{S_t} = \underbrace{\mu dt}_{\text{deterministic trend}} + \underbrace{\sigma dW_t}_{\text{random fluctuations}}$$

This decomposition shows asset price changes as the sum of deterministic growth and random noise.

3. Moments and Log-normality of Geometric Brownian Motion

Lemma 3.1.1 (Moments of Geometric Brownian Motion)

Let $S_t = S_0 \exp(at + \sigma W_t)$, $a = \mu - \sigma^2/2$ with $\mu, \sigma \in \mathbb{R}$, and fixed (deterministic) initial value S_0.

Then:

1. $\mathbb{E}(S_t) = S_0 e^{\mu t}$
2. $\mathbb{E}(S_t^2) = S_0^2 e^{(2\mu+\sigma^2)t}$
3. $\mathrm{Var}(S_t) = S_0^2 e^{2\mu t}\left(e^{\sigma^2 t} - 1\right)$

Proof: (Exercise for the reader.)

Log-normal Distribution: Definition and Properties

Definition 3.1.2 (Log-normal distribution). A vector-valued random variable $X(\omega) \in \mathbb{R}^d$ is **log-normal** if $X = (\ln X_1, ..., \ln X_d)^\top \sim \mathcal{N}(\xi, \Sigma)$ for some $\xi \in \mathbb{R}^d$ and a symmetric, positive definite covariance matrix $\Sigma \in \mathbb{R}^{d\times d}$.

The expectation and covariance matrix have the entries:

$$\begin{aligned}
\mathbb{E}_i(X) &= e^{\xi_i + \frac{1}{2}\Sigma_{ii}} \\
\mathrm{V}(X)_{ij} &= \mathbb{E}\left((X_i - \mathbb{E}_i(X))(X_j - \mathbb{E}_j(X))\right) = e^{\xi_i+\xi_j+\frac{1}{2}(\Sigma_{ii}+\Sigma_{jj})}(e^{\Sigma_{ij}} - 1)
\end{aligned}$$

For $d = 1$ and $\Sigma = \sigma^2$, the corresponding density is:

$$\phi(x;\xi,\sigma) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma x}\exp\left(-\frac{(\ln x - \xi)^2}{2\sigma^2}\right), & x > 0 \\ 0, & \text{else} \end{cases}$$

Proof: (Exercise for the reader.)

Example: The (one-dimensional) geometric Brownian motion

$$S_t = S_0 \exp(at + \sigma W_t)$$

is log-normal, because

$$\ln S_t = \ln S_0 + at + \sigma W_t \sim \mathcal{N}(\ln S_0 + at, \sigma^2 t).$$

4. Derivation of the Black–Scholes Equation

4.1 Replication and Self-financing Portfolio

Situation: S_t is the value of an underlying, B_t value of a bond.

Goal: Determine the fair price V_t of an option.

Replication strategy: Consider a portfolio containing $a_t \in \mathbb{R}$ underlyings and $b_t \in \mathbb{R}$ bonds:

$$V_t = a_t S_t + b_t B_t$$

(cf. 1.5). Assume the portfolio is self-financing: no cash in/out; buying an item must be financed by selling another.

Consequence:

$$V_{t+\delta} - V_t = (a_{t+\delta}S_{t+\delta} - a_t S_t) + (b_{t+\delta}B_{t+\delta} - b_t B_t) \approx a_t(S_{t+\delta} - S_t) + b_t(B_{t+\delta} - B_t)$$

for all $t \geq 0$ and small $\delta > 0$. For $\delta \to 0$ (in an integral sense),

$$dV_t = a_t dS_t + b_t dB_t$$

4.2 Asset Dynamics

Now suppose

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ dB_t &= r B_t dt \end{aligned}$$

with $\mu, r, \sigma \in \mathbb{R}$.

Therefore,

$$dV_t = a_t(\mu S_t dt + \sigma S_t dW_t) + b_t(r B_t dt) = (a_t \mu S_t + b_t r B_t)dt + a_t \sigma S_t dW_t \tag{3.2}$$

4.3 Using Itô's Formula for the Option Price

Assume that the value of the option is a function of t and S_t, i.e. $V_t = V(t, S_t)$. Apply Itô's formula:

$$dV(t, S_t) = \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\mu S_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S_t^2 \right) dt + \frac{\partial V}{\partial S}\sigma S_t dW_t \tag{3.3}$$

Matching dW_t-terms in (3.2) and (3.3):

$$a_t = \frac{\partial V}{\partial S}(t, S_t)$$

Matching dt-terms:

$$
\begin{aligned}
a_t\mu S_t + b_t r B_t &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\mu S_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S_t^2 \\
\Rightarrow \quad b_t r B_t &= \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S_t^2 \\
\Rightarrow \quad b_t &= \frac{1}{rB_t}\left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S_t^2\right)
\end{aligned}
$$

4.4 The Black–Scholes PDE

Plug back:

$$V(t, S_t) = a_t S_t + b_t B_t = \frac{\partial V}{\partial S}(t, S_t)S_t + b_t B_t$$

Multiply by r and rearrange:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

This is the Black–Scholes equation.

4.5 Terminal and Boundary Conditions

Terminal condition: At expiry T,

$$V(T, S) = \psi(S)$$

where $\psi(S)$ is the payoff function, e.g. $\psi(S) = (S-K)^+$ for a call, $\psi(S) = (K-S)^+$ for a put.

Boundary at $S = 0$: $V(t, 0) = 0$ for calls, $V(t, 0) = e^{-r(T-t)}K$ for puts.

Boundary as $S \to \infty$: $V(t, S) \sim S$ for calls.

At $S = 0$, no boundary is needed if V is regular enough:

$$\lim_{S\to 0}\frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}(t, S) = 0, \qquad \lim_{S\to 0} rS\frac{\partial V}{\partial S}(t, S) = 0$$

so $0 = \partial_t V(t, 0) - rV(t, 0) \implies V(t, 0) = e^{-r(T-t)}V(T, 0)$.

Remark:

The parameter μ from the SDE for S_t does **not** appear in the Black–Scholes equation. This is because pricing is risk-neutral and does not depend on the real-world drift.

5. Black–Scholes Formulas

First goal: Solve the Black–Scholes equation for a European call:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \qquad V(T,S) = (S-K)^+$$

with $r, \sigma, T, K > 0$.

Next: In the next section, we will derive the explicit Black–Scholes formula for European calls and puts.

Step 1: Transformation to the heat equation

Define new variables:

$$\begin{aligned} x(S) &= \ln(S/K) & x &: (0,\infty) \longrightarrow (-\infty,\infty) \\ \tau(t) &= \frac{\sigma^2}{2}(T-t) & \tau &: [0,T] \longrightarrow [0,\sigma^2 T/2] \\ w(\tau,x) &= \frac{V(t,S)}{K} & w &: [0,\sigma^2 T/2] \times (-\infty,\infty) \longrightarrow \mathbb{R} \end{aligned}$$

Derivatives in new variables:

$$\begin{aligned} \partial_t V(t,S) &= K\partial_\tau w(\tau,x) = K\partial_\tau w(\tau,x)\frac{d\tau}{dt} = -K\frac{\sigma^2}{2}\partial_\tau w(\tau,x) \\ \partial_S V(t,S) &= K\partial_x w(\tau,x)\frac{dx}{dS} = \frac{K}{S}\partial_x w(\tau,x) \qquad \left(\text{because } \frac{dx}{dS} = \frac{1}{S/K}\cdot\frac{1}{K} = \frac{1}{S}\right) \\ \partial_S^2 V(t,S) &= \cdots = \frac{K}{S^2}\left(\partial_x^2 w(\tau,x) - \partial_x w(\tau,x)\right) \end{aligned}$$

Insert into the Black–Scholes equation:

$$\begin{aligned} 0 &= \partial_t V(t,S) + \frac{\sigma^2}{2}S^2\partial_S^2 V(t,S) + rS\partial_S V(t,S) - rV(t,S) \\ &= -K\frac{\sigma^2}{2}\partial_\tau w(\tau,x) + \frac{\sigma^2}{2}S^2\frac{K}{S^2}\left(\partial_x^2 w(\tau,x) - \partial_x w(\tau,x)\right) \\ &\quad + rS\frac{K}{S}\partial_x w(\tau,x) - rKw(\tau,x) \\ &= -K\frac{\sigma^2}{2}\partial_\tau w(\tau,x) + \frac{\sigma^2}{2}K\left(\partial_x^2 w(\tau,x) - \partial_x w(\tau,x)\right) + rK\partial_x w(\tau,x) - rKw(\tau,x) \end{aligned}$$

Divide by $K\frac{\sigma^2}{2}$:

$$\partial_\tau w(\tau,x) = \partial_x^2 w(\tau,x) - \partial_x w(\tau,x) + c\partial_x w(\tau,x) - cw(\tau,x)$$

with $c := 2r/\sigma^2$.

Next, we eliminate the last three terms. Ansatz:

$$u(\tau, x) = e^{-\alpha x - \beta\tau} w(\tau, x), \qquad \alpha, \beta \in \mathbb{R}$$

Substitute:

$$\begin{aligned}\partial_\tau u(\tau, x) &= -\beta u(\tau, x) + e^{-\alpha x - \beta\tau} \partial_\tau w(\tau, x) \\ &= -\beta u(\tau, x) + e^{-\alpha x - \beta\tau} \left(\partial_x^2 w(\tau, x) - \partial_x w(\tau, x) + c\partial_x w(\tau, x) - cw(\tau, x)\right)\end{aligned}$$

Since

$$\begin{aligned}\partial_x w(\tau, x) &= \partial_x \left(e^{\alpha x + \beta\tau} u(\tau, x)\right) = \alpha e^{\alpha x + \beta\tau} u(\tau, x) + e^{\alpha x + \beta\tau} \partial_x u(\tau, x) \\ \partial_x^2 w(\tau, x) &= \alpha^2 e^{\alpha x + \beta\tau} u(\tau, x) + 2\alpha e^{\alpha x + \beta\tau} \partial_x u(\tau, x) + e^{\alpha x + \beta\tau} \partial_x^2 u(\tau, x)\end{aligned}$$

it follows that

$$\begin{aligned}\partial_\tau u(\tau, x) = &- \beta u(\tau, x) + \left(\alpha^2 u(\tau, x) + 2\alpha\partial_x u(\tau, x) + \partial_x^2 u(\tau, x)\right) \\ &- (\alpha u(\tau, x) + \partial_x u(\tau, x)) + c\left(\alpha u(\tau, x) + \partial_x u(\tau, x)\right) - cu(\tau, x)\end{aligned}$$

or

$$\partial_\tau u(\tau, x) = \partial_x^2 u(\tau, x) + (2\alpha - 1 + c)\,\partial_x u(\tau, x) + \left(\alpha^2 - \alpha + c\alpha - c - \beta\right) u(\tau, x)$$

Hence, the terms including $u(\tau, x)$ and $\partial_x u(\tau, x)$ vanish if

$$-\beta + \alpha^2 + (c-1)\alpha - c = 0 \qquad \text{and} \qquad 2\alpha + (c-1) = 0.$$

The solution is

$$\alpha = -\frac{1}{2}(c-1), \qquad \beta = -\frac{1}{4}(c+1)^2 = -(1-\alpha)^2.$$

With these parameters, $u(\tau, x)$ solves the **heat equation**

$$\partial_\tau u(\tau, x) = \partial_x^2 u(\tau, x), \qquad x \in \mathbb{R},\ \tau \in [0, \sigma^2 T/2]$$

with initial condition

$$u(0, x) = e^{-\alpha x} w(0, x) = e^{-\alpha x} \frac{V(T, S)}{K} = e^{-\alpha x} \frac{(S-K)^+}{K} = e^{-\alpha x}(e^x - 1)^+$$

since $S = Ke^x$.

Step 2: Solving the heat equation

Lemma 3.3.1 (solution of the heat equation)
Let $u_0 : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies the growth condition

$$|u_0(x)| \leq Me^{\gamma x^2}$$

with constants $M > 0$ and $\gamma \in \mathbb{R}$. Then, the function

$$u(\tau, x) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-\xi)^2}{4\tau}\right) u_0(\xi)\, d\xi$$

is the unique solution of the heat equation

$$\partial_\tau u(\tau, x) = \partial_x^2 u(\tau, x), \qquad x \in \mathbb{R},\ \tau > 0$$

and we have

$$\lim_{\tau \to 0} u(\tau, x) = u_0(x).$$

Proof. The fact that u solves the PDE can be checked by substituting and computing the partial derivatives (exercise). The last assertion can be verified via the transformation $\eta = (x - \xi)/\sqrt{4\tau}$ (exercise). Uniqueness follows from the maximum principle.
By a tedious[1] calculation, it can be shown that

$$u(\tau, x) = \exp\left((1-\alpha)x + (1-\alpha)^2\tau\right) \Phi(d_1) - \exp\left(-\alpha x + \alpha^2\tau\right) \Phi(d_2)$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2}\, ds \tag{3.5}$$

$$d_{1/2} = \frac{\ln \frac{S}{K} + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \tag{3.6}$$

Remark: $\Phi(x)$ is the cumulative distribution function of the standard normal distribution.

Step 3: Inverse transform

Since $\beta = -(1-\alpha)^2$ and $\beta + \alpha^2 = 2\alpha - 1 = -c$ it follows that

$$\begin{aligned} V(t,S) &= Ku(\tau,x) = K\exp(\alpha x + \beta\tau)u(\tau,x) \\ &= K\exp(\alpha x + \beta\tau)\exp\left((1-\alpha)x + (1-\alpha)^2\tau\right)\Phi(d_1) \\ &\qquad - K\exp(\alpha x + \beta\tau)\exp\left(-\alpha x + \alpha^2\tau\right)\Phi(d_2) \\ &= \underbrace{K\exp(x)}_{S}\Phi(d_1) - K\exp\left(\underbrace{(\beta+\alpha^2)}_{-c}\tau\right)\Phi(d_2) \\ &= S\Phi(d_1) - K\exp(-r(T-t))\Phi(d_2) \end{aligned}$$

Check boundary:

$$V(t,0) = -K\exp(-r(T-t))\Phi(d_2) = \exp(-r(T-t))V(T,0) \iff (3.4) \quad \checkmark$$

Check terminal condition:

$$V(T,S) = S\Phi(d_1) - K\Phi(d_2)$$

By definition of $d_{1/2} = d_{1/2}(t)$

$$\lim_{t\to T} d_{1/2}(t) = \lim_{t\to T} \frac{\ln\frac{S}{K} + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = \lim_{t\to T}\frac{\ln\frac{S}{K}}{\sigma\sqrt{T-t}} = \begin{cases} +\infty & \text{if } S > K \\ 0 & \text{if } S = K \\ -\infty & \text{if } S < K \end{cases}$$

and hence

$$\lim_{t\to T}\Phi(d_{1/2}(t)) = \begin{cases} 1 & \text{if } S > K \\ 1/2 & \text{if } S = K \\ 0 & \text{if } S < K \end{cases} \implies \lim_{t\to T} V(t,S) = \begin{cases} S-K & \text{if } S > K \\ 0 & \text{if } S = K \\ 0 & \text{if } S < K \end{cases} \quad \checkmark$$

All in all, we have shown the following

Theorem 3.3.2 (Black-Scholes formula for calls) *If $r, \sigma, K, T > 0$, then the **Black-Scholes formula***

$$V(t,S) = S\Phi(d_1) - K\exp(-r(T-t))\Phi(d_2)$$

with Φ and $d_{1/2}$ from (3.5) and (3.6), respectively, is the (unique) solution of the Black-

[01] ... so tedious that we do not even dare to ask the reader to prove this as an exercise.

Scholes equation for European calls, i.e.

$$\partial_t V(t,S) + \frac{\sigma^2}{2} S^2 \partial_S^2 V(t,S) + rS\partial_S V(t,S) - rV(t,S) = 0 \qquad t \in [0,T], \quad S > 0$$

$$V(T,S) = (S-K)^+.$$

Corollary 3.3.3 (Black-Scholes formula for puts) *The Black-Scholes equation for a European put*

$$\partial_t V(t,S) + \frac{\sigma^2}{2} S^2 \partial_S^2 V(t,S) + rS\partial_S V(t,S) - rV(t,S) = 0 \qquad t \in [0,T], \quad S > 0$$

$$V(T,S) = (K-S)^+$$

with $r, \sigma, K, T > 0$ *has the unique solution*

$$V(t,S) = K\exp(-r(T-t))\Phi(-d_2) - S\Phi(-d_1)$$

with Φ *and* $d_{1/2}$ *from (3.5) and (3.6), respectively.*

Proof. Let $V_C(t,S)$ be the value of a call with the same T and K. The put-call-parity (Lemma 1.4.1) and Theorem 3.3.2 imply

$$\begin{aligned} V(t,S) &= e^{-r(T-t)}K + V_C(t,S) - S \\ &= e^{-r(T-t)}K + S\Phi(d_1) - K\exp(-r(T-t))\Phi(d_2) - S \\ &= e^{-r(T-t)}K(1-\Phi(d_2)) + S(\Phi(d_1) - 1) \\ &= e^{-r(T-t)}K\Phi(-d_2) - S\Phi(-d_1) \end{aligned}$$

because $\Phi(x) + \Phi(-x) = 1$.